The Cramer-Rao Bound for Parallel MRI: Optimum SNR, Minimum Error, and the *g*-Function

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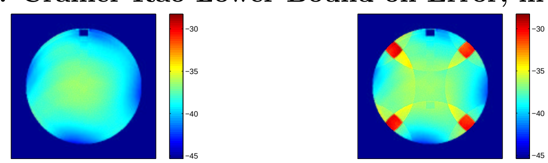
Introduction: Some reasons for the strong appeal of the SENSE algorithm [1] in parallel MR are: it provides theoretically maximum output SNR and also minimum least-squares estimation error, and it also offers a self-assessment of its performance via the g-function. The purpose of this abstract is to explain the close relationship of these properties are to the Cramer-Rao Bound (information inequality), [2], [3]. Essentially, this bound can compare the performance of *any* reconstruction algorithm (image estimator) to the optimal (least-squares = SENSE) estimator. This is a very powerful property. Because it quantifies algorithm performance under very general circumstances, the Cramer-Rao bound is widely used in many branches of electrical engineering and other detection/estimation applications. For example, in industry it is often applied to test whether a given algorithm (*not* necessarily least-squares!) is close to the theoretical limits of its performance. Although used in other medical imaging applications, e.g. [4], the Cramer-Rao bound does not seem to have been considered in the parallel MR literature to date.

What is the Cramer-Rao Bound (CRB)? To give a brief quantitative description of the CRB, we need notation for the least-squares estimator of an N-pixel MR image $\mathbf{x} = (x_1, x_2, \ldots, x_N)$, based on space-domain measurements $\mathbf{z} = (\mathbf{z}^1, \mathbf{z}^2, \ldots, \mathbf{z}^M)$ taken from M receiver coils. In the usual notation of filtering theory [3], the least-squares estimator L of $\mathbf{x}$ would be written $\mathbf{x} = L\mathbf{z} = PH^\dagger R^{-1}\mathbf{z}$. Here, $P = (H^\dagger R^{-1} H)^{-1}$ is the least-squares estimation error covariance, and the measurement model is $\mathbf{z} = H\mathbf{x} + \mathbf{v}$, where measurement noise $\mathbf{v}$ has covariance R. Equivalently, in SENSE notation, $\mathbf{x} = (E^\dagger \Psi^{-1} E)^{-1} E^\dagger \Psi^{-1} \mathbf{z}$, where $E = H$ is the encoding matrix and $\Psi = R$. Suppose F is some other general unbiased, linear estimator– i.e. an MR reconstruction algorithm which need not be the least squares SENSE method– and that the noise is Gaussian. Then the Cramer-Rao bound says the error covariance Σ^F of this other estimator F is bounded below by that of the optimum (least-squares) estimator L, that is, $\Sigma^F \geq P = (E^\dagger \Psi^{-1} E)^{-1}$. This is a comparison of non-negative definite matrices, not a component-by-component bound. More generally, there are non-Gaussian, nonlinear versions of the CRB available, and also alternatives such as the Barankin or Tichavsky-Nehorai bounds. For non-Gaussian problems, P must be replaced by the inverse information matrix. There is not enough space here to include the hypotheses required for the validity of these bounds, but it is important to check them, e.g. they fail for the uniform distribution.

Example: Data was acquired on a 4-coil water phantom using a special-purpose 4-element receiver coil array in our 1.5T GE Signa MR scanner, using a fast spin-echo sequence; a separate off-line acquisition was used to estimate the coil sensitivity functions. Figure (1a) shows the Cramer-Rao lower bound for image reconstruction from fully-sampled data, expressed in dB because the signal-to-noise ratio is very high for this experiment. Just like the g-function, the bound predicts that the greatest errors will occur at the center of the image, farthest from the array elements– this should not be surprising. As an algorithmic challenge, the data was subsampled by a factor of 2 both in the horizontal and vertical directions, which is the fastest rate that will support invertibility of the reconstruction equations. The Cramer-Rao bound now shows that the biggest errors are to be expected in regions with the largest amount of foldover from the original image.

Fig.1: Cramer-Rao Lower Bound on Error, in dB



(a): Slow, Fully-sampled data (b) Fast, aliased data

SNR Performance: A very interesting insight, which is implicit in [1], is that the usual *lower* bound on the estimation error also gives rise to an *upper* bound on output signal-to-noise ratio. For simplicity, we consider the estimate just of pixel x_i, obtained from the ith row of F, denoted $\mathbf{f}_i^\dagger$. Using the fact that the general CRB implies a particular special case for diagonal elements, namely $\Sigma^F_{i,i} = \mathbf{f}_i^\dagger R\, \mathbf{f}_i \geq P_{i,i}$, we obtain

$$\mathrm{SNR}_i^F = \frac{\mathbb{E}\left(\,|\,\langle \mathbf{f}_i, \mathbf{s}\rangle\,|^2\right)}{\mathbf{f}_i^\dagger R\, \mathbf{f}_i} = \frac{|x_i|^2}{\mathbf{f}_i^\dagger R\, \mathbf{f}_i} \leq \mathrm{SNR}_i^L = \frac{|x_i|^2}{P_{i,i}}$$

In the second expression, "$\mathbb{E}$" is the expectation operator (averaging). Using the assumption of **unbiasedness** was crucial in order to obtain the third expression. The unbiased condition is referred to as an orthogonality condition in [1]. Note that maximizing SNR is not always equivalent to minimizing least-squares error. See [5] for a counterexample, and for relationships with the MR phased array [6].

Conclusions: Basically, the g function is a ratio $P^{\mathrm{alias}}_{i,i} / P^{\mathrm{full}}_{i,i}$ that compares performance for aliased and fully-sampled least-squares problems. By contrast, it is not necessary to take ratios in the bounds we have described here, and they can be applied to get insight on performance of other reconstruction methods, which are not least-squares. Like the g-function, these bounds give geometric insight into the spatial distribution of errors.

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